Delta and Gamma hedging of mortality and interest-rate risk

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Outline

GENERAL PROBLEM

Price and hedge life contracts in the presence of systematic mortality risk

- starting from a continuous-time description of stochastic mortality which can be handled analytically and is RELIABLE under the historical measure
- WITHOUT IMPOSING no arbitrage
- with a manageable, PARSIMONIOUS model
- which integrates INTEREST-RATE RISK, still in a PARSIMONIOUS way

SPECIFIC AIM

Obtain in closed-form Delta and Gamma sensitivities and hedges for the reserves.

WHY? in order to

- get an intuitive representation of mortality risk as difference between forecasted and actual mortality intensity
- get an hedge easy to compute and monitor
- easily incorporate budget constraints (linear systems)
- include Delta and Gamma coverage of interest-rate risk
- foster liquidity and develop a secondary market for longevity bonds

SOLUTION

- for each generation, we use an affine stochastic intensity which has the Gompertz law as non-stochastic counterpart (under the historical measure)
- we prove that there exist measure changes which permit to adopt an Heath Jarrow and Morton (HJM) –like framework for pricing/reserving, without imposing no arbitrage
- we characterize prices/reserves and Greeks under such measures
- we solve with both riskless and risky Hull–White interest rates

WHAT ABOUT APPLICATIONS?

As an example we

- calibrate mortality to UK insured males (historical measure)
- calibrate interest rate to the UK Government–bond market (risk neutral measure)
- compute sensitivity and hedges of pure endowments

MORTALITY RISK under historical measure $\mathbb{P}$

- Death arrival is modelled as the first jump time of a doubly stochastic process.
- Let $\lambda_x(t)$ be the mortality intensity of generation x at time t. We assume that

$$d\lambda_x(t) = a(t, \lambda_x(t))dt + \sigma(t, \lambda_x(t))dW_x(t) \tag{1}$$

  with a and σ affine in λ_x (Assumptions 1 and 2)
- Let $S_x(t, T)$ be the probability for a head of generation x, alive at time t, to survive from t to T. Then

$$S_x(t, T) = e^{\alpha(T-t)+\beta(T-t)\lambda_x(t)}$$

  where $\alpha(\cdot)$ and $\beta(\cdot)$ solve appropriate Riccati equations.

MORTALITY RISK II

The forward death intensity is defined as

$$f_x(t,T) = -\frac{\partial}{\partial T}\ln(S_x(t,T)).$$

It represents the probability of dying right after T, as forecasted at t. It is the "best forecast" of the actual one, λ, since

$$f_x(T,T) = \lambda_x(T)$$

TWO SPECIAL CASES

- Ornstein-Uhlenbeck (OU) process without mean reversion

$$d\lambda_x(t) = a\lambda_x(t)dt + \sigma dW_x(t)$$

- Feller (FEL) process without mean reversion

$$d\lambda_x(t) = a\lambda_x(t)dt + \sigma\sqrt{\lambda_x(t)}dW_x(t)$$

WHY?

- conditions for λ_x to be positive and for the survival probability to be decreasing in T are specified and/or verified
- both have the Gompertz law as expectation
- parsimonious models, good for analytical tractability
- although they are very simple, they proved to fit accurately historical and projected mortality tables (Luciano and Vigna, 2008; Luciano, Spreeuw and Vigna, 2008), better than their mean reverting counterparts (*).
- all in all, the requirements for a good mortality model listed by Cairns, Blake and Dowd (2006) seem to be satisfied

(*) for four generations from 1885 to 1945 the m.s.e. with respect to Human Mortality Database and IML92 data range from 0.00012 to 0.00085.

FINANCIAL RISK under historical measure $\mathbb{P}$

- Let $F(t,T)$ be the time-t forward interest rate for maturity T, so that $B(t,T) = \exp\left(-\int_t^T F(t,u)du\right)$.
- We assume that

$$dF(t,T) = A(t,T)dt + \Sigma(t,T)dW_F(t)$$

with W_F independent of all W_x (Assumption 3)

CHANGE OF MEASURE

- Let the SYSTEMATIC MORTALITY RISK premium be

$$\theta_x(t) := \frac{p(t) + q(t)\lambda_x(t)}{\sigma(t, \lambda_x(t))}$$

with $p(t)$ and $q(t)$ continuous functions of time (Assumption 4). There exists an equivalent measure $\mathbb{Q}$ under which λ is still affine

$$\Rightarrow \quad d\lambda_x(t) = [a(t, \lambda_x(t)) + p(t) + q(t)\lambda_x(t)]\, dt + \sigma(t, \lambda_x(t))dW'_x.$$

- For OU and FEL we choose $p = 0$ and $q \in \mathbb{R}$ (constant risk premium), so that the mortality intensity is still OU and FEL.
- This implies that $\mathbb{Q}$ is not only EQUIVALENT, but also RISK NEUTRAL, that is arbitrages are ruled out (Theorem 1).

CHANGE OF MEASURE II

- We assume no risk premium for the IDIOSYNCRATIC MORTALITY RISK (Assumption 5)
- As customary, we assume that no arbitrage holds in the FINANCIAL market. For simplicity, we let the market be complete (Assumption 6). Then

$$dF(t,T) = A'(t,T)dt + \Sigma(t,T)dW'_F(t)$$

where A' satisfies the HJM relationship:

$$A'(t,T) = \Sigma(t,T)\int_t^T \Sigma(t,u)du$$

PRICING/RESERVING CONSEQUENCES

Consider a pure endowment (Arrow Debreu security) with expiration T, on an individual of generation x. Its price – or fair value of the obligation or reserve – is

$$P_x(t,T) = S_x(t,T)B(t,T) = \exp\left(-\int_t^T [f_x(t,u) + F(t,u)]\,du\right)$$

where f_x and F are measure-changed.
Before t, P_x is stochastic: $\tilde{P} = \tilde{S}_x(t,T)\tilde{B}(t,T)$.

MORTALITY RISK EXPOSURE

Under Assumption 4

$$\tilde{S}(t,T) = \frac{S(0,T)}{S(0,t)} \exp\left[-\int_t^T \int_0^z \left[v(u,T)du + w(u,T)dW'(u) \right] dz \right]$$

In the OU case

$$\tilde{S}(t,T) = \frac{S(0,T)}{S(0,t)} \exp\left[-X(t,T)I(t) - Y(t,T) \right]$$

where

$$a' := a + q$$

$$X(t,T) := \frac{\exp(a'(T-t)) - 1}{a'}$$

$$Y(t,T) := -\sigma^2 [1 - e^{-2a't}] X(t,T)^2/(4a')$$

and $I(t)$ is the mortality risk factor or forecast error:

$$I(t) := \tilde{\lambda}(t) - f(0,t)$$

Notice that the hedge depends on the risk premium q, but the risk factor is independent of the horizon of the survival probability, T.

SENSITIVITY to mortality risk

If $F(t,T) = 0$ for all t and T, then $S = P$ and the sensitivity of the reserve to the mortality risk factor is

$$dP = dS = \frac{\partial S}{\partial t} dt + \frac{\partial S}{\partial I} dI + \frac{1}{2} \frac{\partial^2 S}{\partial I^2} (dI)^2$$

In the OU case

$$\Delta^M = \frac{\partial S}{\partial I} = -SX \leq 0$$

$$\Gamma^M = \frac{\partial^2 S}{\partial I^2} = SX^2 \geq 0$$

FINANCIAL RISK EXPOSURE

If $F(t,T)$ satisfies Assumption 3 and is Hull-White under $\mathbb{Q}$, namely

$$\Sigma(t,T) = \Sigma \exp(-g(T-t)), \quad \Sigma > 0, g > 0$$

then

$$\tilde{B}(t,T) = \frac{B(0,T)}{B(0,t)} \exp\left[-\bar{X}(t,T)K(t) - \bar{Y}(t,T)\right]$$

where $\bar{X}$ and $\bar{Y}$ are defined similarly to X and Y of the mortality risk and $K(t)$ is the financial risk factor or forecast error, measured by the difference between the short and forward rate:

$$K(t) := \tilde{r}(t) - F(0,t)$$

SENSITIVITY to mortality and financial risk

If $F(t,T)$ is not identically null, $P = SB$ and

$$dP = BdS + SdB$$

For fixed t

$$dP = B\left[\Delta^M dI + \frac{1}{2}\Gamma^M (dI)^2\right] + S\left[\Delta^F dK + \frac{1}{2}\Gamma^F (dK)^2\right]$$

where

$$\Delta^F = \frac{\partial B}{\partial K} = -B\overline{X} \leq 0$$

$$\Gamma^F = \frac{\partial^2 B}{\partial K^2} = B\overline{X}^2 \geq 0$$

DELTA GAMMA HEDGING

Given n endowments, we can hedge them using m additional hedging contracts with different expiry.

- we build the portfolio

$$\Pi(t) = nP + \sum_{i=1}^{m} n_i P(t, T_i)$$

- Then, the numbers of hedging contracts n_i can be chosen so as to make the portfolio deltas and gammas null (linear systems):

$$\Delta_\Pi^M = \Gamma_\Pi^M = \Delta_\Pi^F = \Gamma_\Pi^F = 0$$

- $n_i < 0$ means a net sale of pure endowments, $n_i > 0$ a net purchase of longevity bonds
- a self-financing portfolio requires $\Pi(0) = 0$
- can be extended to other insurance policies/assets

EXAMPLE

- Take an insurance company which sold n pure endowments with maturity T, i.e. a portfolio short n contracts with value $P(0,T)$.
- They can fix two tenors T_1 and T_2 and choose n_1, n_2 so that the portfolio made up of n, n_1, n_2 endowments/longevity bonds is Delta and Gamma hedged.
- Or they can choose n, n_1, n_2 so that it is self financed and Delta and Gamma hedged.

CALIBRATED EXAMPLE

- we calibrate OU intensity to the survival probabilities of 65-years old UK males using insured data (IML tables), i.e. under the $\mathbb{P}$ measure. The ML estimates are $a = 10.94\%$, $\sigma = 0.07\%$
- we switch from $\mathbb{P}$ to $\mathbb{Q}$ using Assumption 4, which makes the intensity still OU under $\mathbb{Q}$. In this application we select $q = 0$
- we calibrate Hull-White interest rates to UK Government-bond quotes, i.e. under the $\mathbb{Q}$ measure: $g = 2.72\%, \Sigma = 0.65\%$

CALIBRATED EXAMPLE II

For a pure endowment with maturity T, we obtain

Maturity T	Δ^M	Γ^M	Δ^F	Γ^F
5	-6.274	41.757	-4.299	20.096
15	-27.192	1034.084	-6.96	85.721
25	-41.771	5501.92	-4.56	82.713

Notice that $|\Delta^M| > |\Delta^F|$ and $\Gamma^M > \Gamma^F$.

However, under realistic hypothesis on the shocks – or risk factor realizations – ΔI and ΔK the effect of mortality and financial risk have the same order of magnitude, i.e.

$$\Delta^M \Delta I \simeq \Delta^F \Delta K \qquad \text{and} \qquad \Delta^M \Delta I + \frac{1}{2}\Gamma^M \Delta I^2 \simeq \Delta^F \Delta K + \frac{1}{2}\Gamma^F \Delta K^2$$

Take for instance $T = 25$, $\Delta I = -5$ bp, $\Delta K = -50$ bp. Then,

$$\Delta^M \Delta I = 0.0209 \qquad \Delta^F \Delta K = 0.0228$$

CALIBRATED EXAMPLE III

To finish, suppose an insurance company sold a pure endowment expiring in 15 years. It can Delta and Gamma hedge its reserve using pure endowments/longevity bonds, as follows.

Mortality risk

1. purchase 1.1 and 0.26 longevity bonds expiring in 10 and 20 years; cost of the hedge: 0.37
2. purchase 0.48 and 0.60 longevity bonds expiring in 10 and 20 years, issue 0.1 pure endowments with maturity 30 years; this is a self financing strategy

Mortality and financial risk

1. take also a short position in 0.6 zero coupon bonds with maturity 5 and 0.1 long positions in zcbs with maturity 20 years; cost of the hedge: -0.14

CONCLUSIONS

The paper introduces a hedging tool for mortality and interest rate risk that:

- is easy-to-handle
- is based on a reliable mortality model
- is based on a standard interest-rate model
- leads to solving linear systems
- is very well-known and widely used when restricted to financial risk only
- last but not least, it can be extended to other insurance contracts (death assurances, annuities...) and mortality derivatives

EXTENSIONS

- explicit treatment of market incompleteness by recognizing the correspondence between the measure selection and the hedging criterion (see He and Pearson, 1991)
- dynamic assessment of hedge effectiveness (hedging error, as in option pricing)
- two population extension in order to include basis risk (Dahl et al., 2008, Cairns et al. 2011a, 2011b)

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APPENDIX

Theorem

Let λ be a purely diffusive process which satisfies Assumption 4. Let its forward intensity under $\mathbb{Q}$ be

$$df(t,T) = v(t,T)dt + w(t,T)dW'(t).$$

Then, the HJM condition

$$v(t,T) = w(t,T)\int_t^T w(t,s)ds$$

is satisfied if and only if:

$$\frac{\partial m(t,T)}{\partial T} = n(t,t)\frac{\partial n(t,T)}{\partial T}$$

where $m(\cdot)$ and $n(\cdot)$ are the drift and diffusion of $S(t,T)$. This condition is satisfied by the OU and the FEL processes with $p = 0$ and q constant.