Longevity Risk Management in Incomplete Markets using a Least Squares Monte Carlo Approach[1]

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Who wants to live forever?
Who waits forever anyway?
Queen

Perhaps a more appropriate question is: who wants to live old an poor - outliving your assets.

- This is longevity risk: the risk that a population live longer than planned so that not enough money has been saved to fulfill the population's needs!

The are several reasons for why this risk is important:

- The amount of wealth at stake and the systematic underestimation of it.
- The fact that it affects us all: as individuals, governments, and corporations.
- The non-diversifiable nature of this risk and lack of liquid instruments which can be used to manage the risk.

Though several players are in the market it has been argued that the only way to manage this risk is to draw on the capital markets.

1. First of all, we consider modeling directly the longevity index instead of using a "classical" approach:
   - By modelling directly the indices more flexibilty is allowed for.
   - Moreover, instead of considering the entire population attention can be restricted to the "objects" of interest.
2. Next, as a first test of the model we use it to price survivor forwards and survivor swaps:
   - This type of derivatives have been suggested as products which can be used to hedge longevity risk.
   - For these products to be successful though flexible pricing methods are needed.
3. Finally, we explore the use of the longevity index model to price other types of derivatives:
   - To do this we use risk adjusted simulation instead of risk adjusted distributions.
   - We show how to price options, both of European style and American style, which are useful for risk management.

The main findings of the paper are:

1. First of all, when considering the two modeling approaches we find that in the short term they yield similar results on average.
    - However, the tail behavior is different.
    - This leads to different estimates of e.g. the VaR.
2. Next, when it comes to pricing survivor forwards and survivor swaps there are also large differences.
    - For example, the risk part of the price is larger for the Longevity Index models.
3. Finally, when using risk adjusted simulation it is "simple" to price survivor options.
    - The cost of these products is significant.
    - And the American feature is very important.

The data we use is taken directly from the Human Mortality Database (HMD) on their homepage:

- For the time being we consider US females (this will be generalized)
- We consider 4 cohorts: the 65-69, the 70-74, the 75-79, and the 80-84 year old (this could be generalized)
- The survivor rates are constructed as products of appropriately lagged yearly rates:

$$s_{65-69,t} = s_{65,t-4}s_{66,t-3}s_{67,t-2}s_{68,t-1}s_{69,t}.$$

- We consider data from 1950 and onwards (we will include additional data)

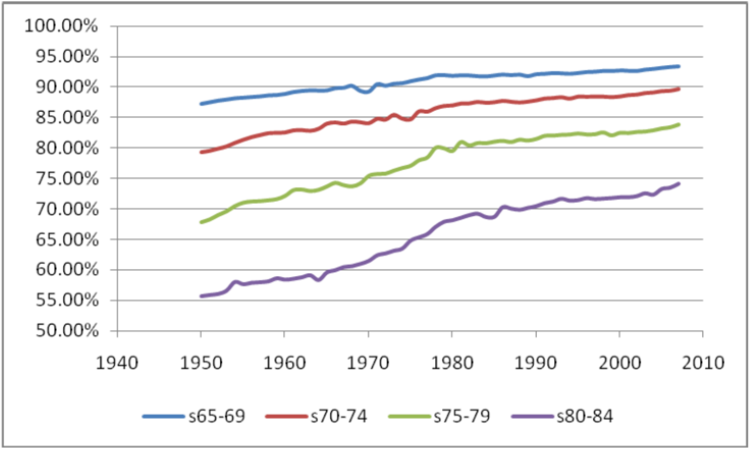


Figure 1: US female cohort survivor rates through time.

We model directly the indices, we refer to this as the LI model, using for the time being only the very simplest possible model.

- Eventually we would like to compare to the Lee & Carter (1992) model so we transform the index as:

$$m_{t,\cdot} = \ln\left(1 - s_{t,\cdot}\right).$$

- We consider a simple AR(1) model given by

$$m_{t,\cdot} = \mu + \beta m_{t-1,\cdot} + \varepsilon_t,$$

  where $\varepsilon_t \sim N\left(0, \sigma^2\right)$.

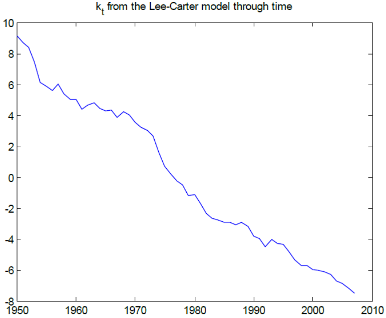
Estimation is performed using simple OLS and we obtain:

- $\mu = -0.0542$ (0.0436) and $\beta = 0.9822$ (0.0181)
- The estimated value for $\sigma = 0.02472$ and the $R^2 = 98.2\%$

As mentioned we will consider the Lee Carter model as a potential benchmark which in our notation corresponds to

$$\ln\left(1 - s_{x,t}\right) = a_x + b_x k_t + \varepsilon_{x,t}.$$

- The time varying element in this model is k_t which is plotted below:



Again we limit attention to a very simple AR(1) model given by

$$k_t = \mu_k + \beta_k k_{t-1} + \varepsilon,$$

where $\varepsilon_t \sim N\left(0, \sigma^2\right)$.
The estimation leads to the following results

- $\mu = 0.2903$ (0.0440) and $\beta = 0.9868$ (0.0094)
- The estimated value for $\sigma = 0.33954$ and the $R^2 = 99.5\%$

The two approaches are thus fundamentally different:

- In the Lee Carter model only k_t is time varying - in the LI model each rate is time varying allowing for more flexibility.
- For forecasting all rates can be predicted from k_t in the Lee Carter model - in the LI model each individual rate is predicted.

Since our metric essentially is one of prediction we now compare the two approaches based on this. We consider simulated survivor rates for the s65-69 cohort in 2015.

- The LI model produces directly predictions of the survivor rates.
- Survivor rates can be generated from the LC model using the appropriate values for a_x and b_x.
- The following table shows important percentiles of the simulated distribution (based on $100,000$ paths).

Table 1: Percentiles of the simulated survivor index in 2015 from the Longevity Index, LI, and the Lee-Carter, LC, model.

quantile	0.01	0.025	0.05	0.1	0.5	0.9	0.95	0.975	0.99
LI	0.9288	0.9301	0.9311	0.9323	0.9364	0.9403	0.9413	0.9422	0.9433
LC	0.9334	0.9339	0.9343	0.9347	0.9362	0.9377	0.9381	0.9385	0.9389

To model several cohort survivor rates we use a simple VAR(1) model. We consider jointly the four series above and for the system we obtain:

Table 3: Parameter estimates and standard errors for VAR(1) model for the four cohort survivor rates.

Const	Std Err	b1,j	Std Err	b2,j	Std Err	b3,j	Std Err	b4,j	Std Err
-0.35817	(0.1334)	0.72023	(0.1310)	-0.07914	(0.1528)	0.39642	(0.1890)	-0.11244	(0.1560)
-0.06215	(0.1044)	0.18676	(0.1026)	0.57606	(0.1197)	0.22737	(0.1480)	-0.01188	(0.1221)
0.10914	(0.1050)	0.14037	(0.1032)	0.04559	(0.1203)	0.64664	(0.1488)	0.21462	(0.1228)
0.08420	(0.0695)	0.11356	(0.0683)	-0.12527	(0.0797)	0.21974	(0.0985)	0.75159	(0.0813)

The $R^2 = 99.9\%$ using the likelihood ratio method.

Comparing to the benchmark model yields:

Table 4: Percentiles of the simulated distribution at future times for the longevity index and Lee & Carter models.

Longevity index model									
quantile	0.01	0.025	0.05	0.1	0.5	0.9	0.95	0.975	0.99
2015	0.9289	0.9300	0.9309	0.9320	0.9355	0.9389	0.9398	0.9406	0.9414
2020	0.8324	0.8347	0.8368	0.8391	0.8470	0.8545	0.8565	0.8583	0.8603
2025	0.7039	0.7084	0.7121	0.7164	0.7308	0.7446	0.7484	0.7514	0.7553
2030	0.5279	0.5345	0.5400	0.5462	0.5679	0.5886	0.5944	0.5992	0.6047
Lee & Carter model									
quantile	0.01	0.025	0.05	0.1	0.5	0.9	0.95	0.975	0.99
2015	0.9334	0.9339	0.9343	0.9347	0.9362	0.9377	0.9381	0.9385	0.9389
2020	0.8390	0.8405	0.8417	0.8432	0.8482	0.8530	0.8544	0.8556	0.8569
2025	0.7120	0.7153	0.7180	0.7211	0.7321	0.7425	0.7455	0.7481	0.7509
2030	0.5382	0.5432	0.5476	0.5526	0.5701	0.5869	0.5917	0.5959	0.6005

The survivor forward involves the exchange of the realized rate of a population at some future date in return for the fixed rate agreed on at the time the contract is entered into. Often the agreed on rate is relative to some common expected rate.

- The payment from a forward is given by

$$s^r_{x,t_1,t_2} - (1+\pi)\, s^e_{x,t_1,t_2},$$

  where s^r_{x,t_1,t_2} is the realized rate, s^e_{x,t_1,t_2} is the expected rate, and π is the price or premium.
- The "fair" price of the swap is fixed such that at the time the contract is entered into the two "legs" have the same net present value

$$\pi = \frac{NPV\left(s^r_{x,t_1,t_2}\right)}{NPV\left(s^e_{x,t_1,t_2}\right)} - 1.$$

The main challenge in valuing survivor derivatives is that the future realized rates are by construction unknown and investors will generally then require a compensation for exposure to these.

- Though several methods exist we choose to use the multivariate extension of the Wang transform which specifies the risk adjusted cumulative distribution for the risk X_i as

$$F^*_{X_i}(x) = \Phi\left[\Phi^{-1}\left(F_{X_i}(x)\right) - \sum_{j=1} \rho_{X_i,X_j}\lambda_i\right],$$

  where λ controls the magnitude of the risk adjustment.
- This requires knowing the correlation structure. However, this is available from the simulation.

Table 5: Survivor forward premium for Lee Carter and LI model for various risk premia and maturities.

Survivor Forward premium										
Lee Carter Model						LI Model				
Lambda	2015	2020	2025	2030		Lambda	2015	2020	2025	2030
0	-0.64	-1.12	-1.10	-2.16		0	-0.72	-1.28	-1.28	-2.56
0.1	-0.62	-1.08	-0.99	-1.93		0.1	-0.69	-1.21	-1.13	-2.27
0.2	-0.61	-1.04	-0.88	-1.70		0.2	-0.66	-1.14	-0.99	-1.99
0.3	-0.60	-0.99	-0.77	-1.48		0.3	-0.63	-1.07	-0.84	-1.71

Table 6: Risk part of the price for the survivor forwards in Table 3

Risk part of premium										
Lee Carter Model						LI Model				
Lambda	2015	2020	2025	2030		Lambda	2015	2020	2025	2030
0.1	0.01	0.05	0.11	0.23		0.1	0.03	0.07	0.15	0.28
0.2	0.02	0.09	0.22	0.46		0.2	0.06	0.14	0.30	0.56
0.3	0.04	0.13	0.34	0.69		0.3	0.09	0.21	0.44	0.85

Table 7: This table shows the survivor swap premium for Lee Carter and LI model for various values of the risk premium and for different maturities.

Survivor Swap premium										
Lee Carter Model						LI Model				
Lambda	2015	2020	2025	2030		Lambda	2015	2020	2025	2030
0	-0.64	-0.85	-0.91	-1.09		0	-0.72	-0.96	-1.04	-1.26
0.1	-0.60	-0.76	-0.75	-0.84		0.1	-0.63	-0.81	-0.80	-0.91
0.2	-0.56	-0.67	-0.58	-0.59		0.2	-0.54	-0.65	-0.55	-0.57
0.3	-0.52	-0.58	-0.41	-0.34		0.3	-0.46	-0.50	-0.31	-0.24

Table 8: This table shows the risk part of the price for the survivor swaps in Table 7.

Risk part of premium										
Lee Carter Model						LI Model				
Lambda	2015	2020	2025	2030		Lambda	2015	2020	2025	2030
0.1	0.04	0.09	0.17	0.25		0.1	0.09	0.16	0.25	0.34
0.2	0.08	0.18	0.33	0.51		0.2	0.18	0.31	0.49	0.68
0.3	0.11	0.27	0.50	0.76		0.3	0.26	0.46	0.73	1.02

In theory it is possible to replicate the survivor swap with a portfolio of forwards and the two portfolios should have the same present value. The following table examines this claim:

Differences										
Lee Carter Model						LI Model				
Lambda	2015	2020	2025	2030		Lambda	2015	2020	2025	2030
0	0.00	0.00	0.00	0.00		0	0.00	0.00	0.00	0.00
0.1	0.02	0.09	0.23	0.40		0.1	0.05	0.16	0.34	0.54
0.2	0.04	0.19	0.46	0.81		0.2	0.10	0.32	0.67	1.08
0.3	0.06	0.28	0.69	1.20		0.3	0.14	0.47	0.99	1.61

The table shows that there are differences:

- These reflect that forwards are priced with "univariate" risk adjustment whereas swaps are priced with "multivariate" risk adjustment.
- Thus, the differences show the effect of neglecting the correlation structure.

Until now we have used risk adjusted distributions. However, in some situations it could be preferred to adjust directly in the simulation:

- Performing risk adjusted simulation means that all paths have equal probability.
- This is convenient for pricing some type of derivatives such as options.
- A potential drawback is that one may need to re-simulate if the risk parameter changes.

The transform used above preserves the normal and the lognormal distributions. Thus, an alternative to the method of adjusting the distribution is to simulate risk adjusted innovations directly.

- To do this we simulate with innovations which are $\varepsilon \sim N\left(-\lambda\sigma, \sigma^2\right)$ instead of $\varepsilon \sim N\left(0, \sigma^2\right)$.
- The differences in distribution are shown below:

quantile	0.01	0.025	0.05	0.1	0.5	0.9	0.95	0.975	0.99
λ=0.3	0.9291	0.9305	0.9316	0.9329	0.9373	0.9415	0.9426	0.9435	0.9447
λ=0.0	0.9279	0.9294	0.9305	0.9318	0.9363	0.9405	0.9417	0.9426	0.9438

A European call option gives the holder the right but not the obligation to “buy” an asset for a given price, called the strike price, at a future time, called the maturity time.

- In the situation where the asset is a survivor rate the option holder would receive the survivor rate while paying the strike price.
- As such the survivor call option is similar to a survivor forward.
- The price of the survivor call option is the present value of the expected payoff at maturity under the risk adjusted measure.
- In our setting this can be estimated by

$$C_{EU} = \frac{1}{N}\sum_{n=1}^{N} \exp\left(-rT\right) \max\left[s_T^n - k, 0\right].$$

European call prices:

	Strike price					
lambda	0.93049	0.93182	0.93633	0.94054	0.94166	0.94220
0	0.50131	0.39517	0.11272	0.01255	0.00557	0.00362
0.1	0.52905	0.42143	0.12726	0.01544	0.00702	0.00462
0.2	0.55693	0.44802	0.14288	0.01886	0.00878	0.00586
0.3	0.58489	0.47490	0.15959	0.02285	0.01091	0.00737

European put prices:

	Strike price					
lambda	0.93049	0.93182	0.93633	0.94054	0.94166	0.94220
0	0.00667	0.01500	0.12073	0.38292	0.47233	0.51686
0.1	0.00521	0.01206	0.10608	0.35661	0.44459	0.48867
0.2	0.00405	0.00961	0.09266	0.33099	0.41731	0.46086
0.3	0.00312	0.00760	0.08048	0.30609	0.39056	0.43349

Recently, simulation methods have also been use to price options with early exercise:

- The problem with this type of derivative is the need to simultaneously determine the optimal stopping strategy.

We use the Least Squares Monte Carlo method of Longstaff & Schwartz (2001):

- At each early exercise the value of holding to option for one more period is determined through a cross-sectional regression.
- In the regression future "cash flows" are regressed on "state variables".
- In our setting the state variable is the survivor rates.

American put prices:

	Strike price					
lambda	0.93049	0.93182	0.93633	0.94054	0.94166	0.94220
0	0.00782	0.01814	0.17427	0.54612	0.65476	0.70716
0.1	0.00632	0.01531	0.16602	0.53911	0.64778	0.70018
0.2	0.00511	0.01285	0.15823	0.53212	0.64081	0.69321
0.3	0.00413	0.01069	0.15080	0.52516	0.63384	0.68625

Early exercise ratio:

	Strike price					
lambda	0.93049	0.93182	0.93633	0.94054	0.94166	0.94220
0	14.68%	17.29%	30.72%	29.88%	27.86%	26.91%
0.1	17.61%	21.23%	36.11%	33.85%	31.37%	30.21%
0.2	20.80%	25.16%	41.44%	37.80%	34.88%	33.52%
0.3	24.44%	28.87%	46.63%	41.72%	38.38%	36.83%

The goal of this research is threefold:

1. We consider modeling directly the longevity index instead of a more "classical" approach.
    - We find that in the short term the two approaches yield similar results on average though the tail behavior is different.
2. We use the models to price survivor forwards and survivor swaps.
    - Our results show, among other things, that the risk part of the price is larger for the Longevity Index models.
3. We explore the use of risk adjusted simulation instead of risk adjusted distributions.
    - We show how to price options, both of European style and American style, which are potentially useful for risk management.