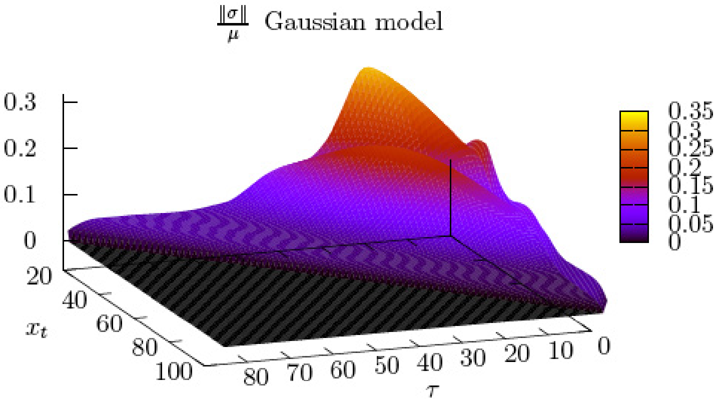

Modeling the Forward Surface of Mortality

D. Bauer, F.-E. Benth & R. Kiesel

"Neoclassical" Stochastic Mortality Setup

(see Biffis, Denuit & Devolder (2009) for more details)

- τ_x is the *time of death* for an x-year old (now $\leftrightarrow$ time zero), inaccessible stopping time, time horizon T^*

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{}_{T_2-T_1}P_{x+T_1}(T_2) &= \mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\{\tau_x > T_2\}} \middle| \mathcal{F}_{T^*} \vee \mathbf{1}_{\{\tau_x > T_1\}}\right] \\
&= \exp\left\{-\int_{T_1}^{T_2} \mu_s(x)\, ds\right\}
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\overset{\text{in life table}}{\longrightarrow}\quad {}_{T_2-T_1}p_{x+T_1}(t;T_2) &= \mathbb{E}^{\mathbb{P}}\left[{}_{T_2-T_1}P_{x+T_1}(T_2) \middle| \mathcal{F}_t\right] \\
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- <u>Observations</u>:
  - Object in **two** dimensions, "age/term" structure $\rightarrow$ generational life table
  - $\left({}_{T_2-T_1}p_{x+T_1}(t;T_2)\right)_{t\geq 0}$ martingale
  - For the ${}_{\cdot}P_{\cdot}$'s, things are like in the "classical" LifeCon setup

  $\Rightarrow$ CLT works under $\mathcal{F}_{T^*}$, so we can disregard "small sample risk" ("unsystematic mortality risk") for most applications

  $\longrightarrow$ Focus on systematic part!

Forward Mortality Setup

(idea and first study by Cairns, Blake & Dowd (2006,ASTIN))

- **Forward force of mortality**:

$$\mu_t(T,x) = \frac{\partial}{\partial T} \log \left\{ {}_{T-t}p_{x+t}(t;T) \right\},\ 0 \leq t \leq T,\ x \geq -t$$

- **Model equation:** $d\mu_t(T,x) = \alpha(t,T,x)\,dt + \sigma(t,T,x)\,dW_t$
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Drift Condition

$$\alpha(t,T,x) = \sigma(t,T,x) \times \int_t^T \sigma(t,s,x)'\,ds$$

- No arbitrage arguments, but martingale property!
  $\rightarrow$ Difference to interest rate theory
- No "life market" assumed, consistency of "best-estimate tables"
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Proposition

$${}_Tp_x(0;T) = {}_sp_x(0;s) \times {}_{T-s}p_{x+s}(0;T) \Longrightarrow \sigma(t,T,x) \equiv 0$$

- Difference to classical "LifeCon calculus": factorization does not hold!

Valuation (see also Bauer, Börger & Ruß (2009,IME) for an application)

- Changing from $\mathbb{P}$ to $\mathbb{Q}$...
  - **Problem**: $\mu_\cdot(\cdot,\cdot)$, $\sigma(\cdot,\cdot,\cdot)$ depend on $\mathbb{P}$, different objects!
  - $\longrightarrow$ Best estimate vs. valuation tables $\rightarrow$ Difference to interest rate theory
- $\longrightarrow$ ...but the $\mu_t(x)$'s coincide. Need to change to spot modeling:

$$d\mu_t(x) = \left(\frac{\partial}{\partial T}\mu_t(T,x)\Big|_{T=t} + \alpha(t,t,x)\,dt \right)\,dt + \sigma(t,t,x)\,dW_t$$

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- However, for **Gaussian models**...

Proposition

If $\sigma(t,T,x)$ and market price of risk $\lambda(t)$ deterministic:

- $\mathbb{E}^{\mathbb{Q}}\left[{}_{T-t}P_{x+t}(T)\middle|\,\mathcal{F}_t\right] = e^{-\int_t^T \int_t^s \sigma(u,s,x)\,\lambda(u)\,du\,ds}\,{}_{T-t}p_{x+t}(t;T)$
- $\sigma(t,T,x)^{\mathbb{P}} = \sigma(t,T,x)^{\mathbb{Q}}$

- To operationalize:
  - Estimate volatility under $\mathbb{P}$ – for pricing mortality-contingent claims, it is now solely necessary to specify risk-adjusted mortality surface – e.g. by Wang transform or simply... (cf. Delbaen & Schachermayer (1994,MathAnn))

.. by "replacing" the mortality table with a "table reflecting a lower mortality rate" which is "common practice in actuarial science"

Consistency of Forward-Factor Models

(see Filipović (2001) for interest rate models)

- LIFE two weeks ago: practitioners started building "simple" forward models to avoid "nested simulations" in their valuation / risk analysis
- **Idea**: (forward Gompertz model)

$$\mu_t(T,x) = Z_t^{(1)} \times \exp\left\{Z_t^{(2)} \times (x+T)\right\} (= G(T-t, x+t, Z_t))$$

- **Question:** Is this an "appropriate" model?

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- **Question:** Is this an "appropriate" model?

Proposition

Diffusion Z with drift χ and volatility ρ is consistent with G iff we have

$$\begin{aligned} G_1(\tau, x+t, Z_t) \quad = \quad & G_2(\tau, x+t, Z_t) + \sum_{i=1}^{m} \chi_{i,t} \frac{\partial}{\partial z_i} G(\tau, x+t, Z_t) + \sum_{i,j=1}^{m} \left(\left(\sum_{k=1}^{d} \rho_{ik,t}\, \rho_{jk,t} \right) \right. \\ & \left. \left(\frac{1}{2} \frac{\partial^2}{\partial z_i \partial z_j} G(\tau, x+t, Z_t) - \frac{\partial}{\partial z_i} G(\tau, x+t, Z_t) \times \int_0^{\tau} \frac{\partial}{\partial z_j} G(u, x+t, Z_t)\, du \right) \right) \end{aligned}$$

- **Answer:** There is no non-trivial diffusion consistent with the forward Gompertz model

Infinite-Dimensional Formulation

(see Musiela (1999) and Filipović (2001) for interest rate models)

- **Idea**: μ, σ, etc. are now elements of a suitable function space. Then formulate dynamics of surface
- **Problem**: The domains are different over time $\rightarrow$ need to change parametrization

$$\bar{\mu}_t(\tau, x_t) = \mu_t\,(t+\tau, x_t - t),\ \bar{\sigma}_t(\tau, x_t) = \sigma\,(t, t+\tau, x_t - t),\ldots$$

- **What space?** (does that matter???)
  - $\longrightarrow$ Hilbert space H of cont. functions
  - $\longrightarrow$ evaluation functional continuous (convergence in H implies point-wise convergence)
  - $\longrightarrow$ There exists a C_0-semigroup $\{S_t\}_{t\geq 0}$ with infinitesimal generator A such that

$$(S_t f)\,(\tau, x) = f(\tau + t, x - t),\ 0 \leq t \leq x$$

We provide examples in the paper: Sobolev-type spaces

$\longrightarrow$ **Model equation**:

$$d\bar{\mu}_t = (A\bar{\mu}_t + \bar{\alpha}_t)\ dt + \sum_{i=1}^{d} \bar{\sigma}_t^{(i)}\, dW_t^{(i)}$$

Key Difference to Interest Rate Modeling: What about $(S_t f)(\tau, x)$ for $x < t$?

- Future generations – not included in "initial" surface!
- We need to make an assumption about future generations!

$\rightarrow$ $\{S_t\}_{t \geq 0}$ becomes a "degree of freedom" for mortality models

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$\rightarrow$ $\{S_t\}_{t\geq 0}$ becomes a "degree of freedom" for mortality models

- If...
  ... H is a space of differentiable functions (real analytic!), S_t is uniquely determined as $S_t = \exp\{A \times t\}$, where $A = \frac{\partial}{\partial \tau} - \frac{\partial}{\partial x}$
  ... H is a space where "kinks" are allowed (first-order Sobolev space), we have modeling freedom. For example,

$$(S_t f)(\tau, x) = f(\tau + x, 0)$$

  $\rightarrow$ Future generations enter the world just as generations today – no systematic improvements!

$\Rightarrow$ The space matters! "Real" consequences for modeling choices. See e.g. below for factor models!

Finite-Dimensional Realizations for Gaussian Models

(see Björk & Gombani (1999,FinStoch) for interest rate models)

- **Goal**: We want to realize the infinite dimensional system by a finite-dimensional realization (FDR)

$$\begin{cases} dZ_t = a(Z_t)\,dt + b(Z_t)\,dW_t,\ Z_0 = 0 \\ \bar{\mu}(\tau, x) = G(\tau, x, Z_t) \end{cases}$$

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Proposition

- A FDR exists iff $\bar{\sigma}(\tau, x) = C(x+\tau) \times \exp\{M\tau\} \times N$
- If H is a space of real-analytic functions, the FDR is given by

$$\begin{cases} dZ_t &= M\,Z_t\,dt + N\,dW_t,\ Z_0 = 0 \\ \bar{\mu}(\tau, x) &= \xi^{\{S_t\}}(t, \tau, x) + C(x+\tau)\,\exp\{M\tau\}\,Z_t \end{cases}$$

- If $\{S_t\}_{t\geq 0}$ is chosen as above ("new generations enter the same"), FDR is given by

$$\begin{cases} dZ_t &= M\,Z_t\,dt + N\,dW_t,\ Z_0 = 0 \\ \bar{\mu}(\tau, x) &= \xi^{\{S_t\}}(t, \tau, x) + C(x+\tau)\,\exp\{M\tau\}\,\left(Z_t - Z_{(t-x)\vee 0}\right) \end{cases}$$

Applications

- Already pointed out :
  - Valuation without nested simulations
  - Guidance on how to build models / check consistency

- Asset Liability Management of Life Insurer:
  - **Question** of how to apply "risk factors" to liability side

  $\longrightarrow$ Need to consistently extrapolate generational life table underlying reserve calculations etc.
  - (Only) answers are given via FDR:

$$
\begin{aligned}
& {}_{T-t}p_{x_t}(t; T-t) \\
= \quad & \exp\left\{-\int_0^{T-t} \bar{\mu}_t(s, x_t)\, ds\right\} \\
= \quad & F\left(\bar{\sigma}, .p.(0; \cdot)\right) \times \left(\exp\left\{\int_0^{T-t} C(x+s)\, e^{M\,s}\, ds\right\}\right)^{-Z_t}
\end{aligned}
$$

$\longrightarrow$ Z_t Normal distributed – easy to simulate! Just fix M, N, C!

Conclusion

- Thorough disquisition of forward mortality models driven by finite-dimensional Brownian motion
- Example in the paper illustrating all the results
- There are key differences to interest rate modeling:
  - Additional dimension "age" $\rightarrow$ instead of *curves*, we have *surfaces*. Require different spaces
  - Here we do not rely on arbitrage arguments, but on martingale properties – no "market" necessary
  - Now we have different surfaces corresponding to different measures – best estimates vs. valuation tables/surfaces
  - New generations are born, which are not considered in "current" surface
    - $\rightarrow$ Semigroup in infinite-dimensional formulation now is part of the model, discretion of the "modeler"
    - Choice of space matters now – consequences for FDR's
- The paper is fairly mathematical, but (we hope that) we demonstrate that our results have direct implications for and applications in practice

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Thank you!